AD-A171 027 ON THE MAINTENANCE OF SYSTEMS COMPOSED OF HIGHLY RELIABLE COMPONENTS..(U) STATE UNIV OF NEW YORK AT STONY BROOK DEPT OF APPLIED MATHEMA.. 1/1

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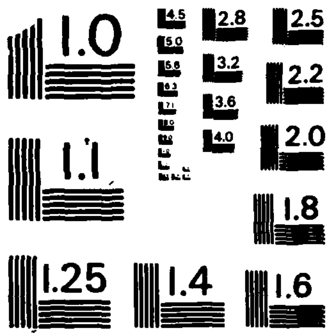

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) AMS-85-57	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 86-0520
6a. NAME OF PERFORMING ORGANIZATION S.U.N.Y. at Stony Brook 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) Dept. of Applied Mathematics and Statistics S.U.N.Y., Stony Brook, N.Y.11794	7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-84-0136
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO.: 61102F PROJECT NO.: 2304 TASK NO.: A5 WORK UNIT NO.:

11. TITLE (Include Security Classification)
ON THE MAINTENANCE OF SYSTEMS COMPOSED OF HIGHLY RELIABLE COMPONENTS

12. PERSONAL AUTHOR(S)
M. N. Katehakis and C. Derman

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Research Report	FROM 7/84 TO 9/85	1985, September, 12	16

16. SUPPLEMENTARY NOTATION
Also supported by the N.S.F. under Grant DMS-84-05413

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB GR.	Reliability Theory, Dynamic Programming, Maintenance.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

We consider the dynamic repair allocation problem for a general multi-component system that is maintained by a limited number of repairmen. Component functioning and repair times are assumed to be exponentially distributed with parameters that may depend on the component but not on repairmen. At most one repairman may be assigned to a failed component. The objective is to determine repair allocation policies that maximize a measure of performance for the system such as the expected discounted system operation time or the availability of the system. We consider systems composed of highly reliable i.e., small failure rates, components and study asymptotic techniques for the determination of optimal policies. In the final section we find asypmptotically optimal policies for the series, parallel and a system composed of parallel subsystems connected in series.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
Brian Woodruff	(202) 767- 5027	NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE. UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

ON THE MAINTENANCE OF SYSTEMS COMPOSED OF HIGHLY RELIABLE COMPONENTS*

by

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ABSTRACT

We consider the dynamic repair allocation problem for a general multi-component system that is maintained by a limited number of repairmen. Component functioning and repair times are assumed to be exponentially distributed with parameters that may depend on the component but not on repairmen. At most one repairman may be assigned to a failed component. The objective is to determine repair allocation policies that maximize a measure of performance for the system such as the expected discounted system operation time or the availability of the system. We consider systems composed of highly reliable i.e., small failure rates, components and study asymptotic techniques for the determination of optimal policies. In the final section we find asypmptotically optimal policies for the series, parallel and a system composed of parallel subsystems connected in series.

AMS 1980 subject classification. Primary: 90B22
OR/MS Index 1981 subject classification. Primary: 722 Reliability, secondary: 116 Dynamic Programming.
Key Words: First passage times, Markov Decision Processes.

*Work supported by USAF Contract AFOSR 840136 and NSF Grant DMS-84-05413.

1. Introduction. Consider an N component system, of known structure, that is maintained by a limited number of R repairmen, i.e., $R < N$. The objective is to characterize dynamic maintenance policies that yield a maximum value to a system measure of performance such as the expected discounted system operation time and the average expected system operation time, or availability of the system. We make the following assumptions. Each component and the system as a whole can be in only two states, functioning or failed. The functioning and repair times for the i^{th} component are exponentially distributed random variables with known parameters μ_i and λ_i. Components are independent i.e., failure or repair of one has no effect on the others. At most one repairman may be assigned to a failed component and it is possible to reassign a repairman from one failed component to another instantaneously. Repaired components are as good as new and failures may take place even while the system is not functioning.

Since the number of available repairmen is less than the number of components, the performance of the system will depend on the maintenance policy employed, i.e., the rule for choosing on which failed components repairmen are assigned to whenever the number of failed components is greater than R. Under these assumptions optimal policies can be obtained, in principle, using methods from Markovian decision theory. However, the computational difficulties are prohibitive due to the very large number of possible states. Therefore, explicit solutions and approximations can provide valuable insight. An explicit solution has been obtained for the series system with N components maintained by a single repairman in Katehakis and Derman (1984); see also Derman et al, (1978), Nash and Weber (1982) and Smith (1978).

In practice many systems are composed of highly reliable components. We

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have a simple model for such systems if we assume that the failure rate for the i^{th} component is of the form $\rho\mu_i$, $1 \leq i \leq N$. Thus, for small values of ρ , all components are highly reliable. We obtain analytical characterizations and derive simple recursive formulas for the determination of policies that are optimal for small values of ρ . These asymptotically optimal policies can be used as good approximations to optimal policies for systems composed of highly reliable components.

The first study of such a model was done in Smith (1978). We extend the work of Smith in the following directions. i) We provide a formulation of the problem along the lines of Markovian decision theory. ii) We treat the multirepairmen case. iii) We note that the recursive formulas for the determination of asymptotically optimal policies essentially constitute a Gauss succesive approximations method for solving the general Markovian decison problem. iv) We establish the existence of intervals of the form $(0,\rho^*)$ with the property that if they contain the failure rates of all components, then the asymptotically optimal policies under consideration are optimal, and v) in the final section we find asypmptotically optimal policies for the series, parallel and a system composed of parallel subsystems connected in series. For the series system maintained by R , $R \geq 2$, repairmen it is asymptotically optimal to assign repairmen to failed components with the longest expected repair times first. Thus, we show that the series system result established in Katehakis and Derman (1984) does not hold in the case of more than one repairmen.

2. Problem Formulation.

Under the assumptions made, at any time the status of all components is given by a vector $x = (x_1,\ldots,x_N)$ with $x_i = 1$ or 0 if the i^{th} component is functioning or failed. Thus $S = \{0,1\}^N$ is the set of all possible states. The structure of the system, i.e., the relation between the

status of the components and that of the system, is given by a partition of the state space S into two sets G and B of "good" and "bad" states; where if $x \in G$ the system is functioning and if $x \in B$ the system is failed. Alternatively, this relation can be specified by the *structure function* ϕ defined on S, such that $\phi(x) = 1$ or 0 if $x \in G$ or $x \in B$.

It is easy to show, using well known results of Markovian decision theory, that the only relevant decision epochs are component failure and repair completion instants and that it suffices to consider only the class of deterministic policies that never leave repairmen idle while there are some failed components. A policy is called *deterministic* if it assigns repairmen to failed components as a deterministic function of the state of the system only. Let Π denote this finite set of deterministic policies that never leave repairmen idle while there are some failed components. The information pattern used by the controller at decision epochs is the state of the system at those instants. When a policy $\pi \in \Pi$ is employed the time evolution of the state of the system can be described by a continuous time, finite state, irreducible, Markov chain $\{x^\pi(t) = (x_1^\pi(t),\ldots,x_N^\pi(t)),\ t \geq 0\}$, where $x_i^\pi(t) = x_i$ if the i^{th} component is in state x_i at time t. Thus, when the initial state of the system is x and a policy $\pi \in \Pi$ is employed, the average expected system operation time and the expected discounted system operation time, for a given discount rate β, are defined by (1) and (2), respectively

(1) $$A_\pi(x) = \lim_{T\to\infty} \frac{1}{T}\, E\left[\int_0^T \phi(x^\pi(t))\, dt \,/\, x^\pi(0) = x\right],$$

(2) $$D_\pi(x,\beta) = E\left[\int_0^\infty e^{-\beta t}\phi(x^\pi(t))\, dt \,/\, x^\pi(0) = x\right].$$

Notice that, for $\pi \in \Pi$, $\{x^\pi(t)\ ,\ t \geq 0\}$ is ergodic thus, we have

(3) $$A_\pi(x) = A(\pi) = \sum_{x \in G} e_\pi(x) ,$$

where $\{e_\pi(x),\ x \in \{0,1\}^N\}$ is the set of ergodic probabilities of $\{x^\pi(t),\ t \geq 0\}$.

It will be convenient to work with the quantities $U_\pi(x,\beta)$ which denote the expected discounted time that the system does not function when the initial state is x and policy π is used. Let $\bar{\phi}(x) = 1 - \phi(x)$ then, $U_\pi(x,\beta)$ is given by (4) below.

(4) $$U_\pi(x,\beta) = E\left[\int_0^\infty e^{-\beta t}\,\bar{\phi}\,(x^\pi(t))\,dt \,/\, x^\pi(0) = x\right]$$

It is easy to see that $D_\pi(x,\beta) + U_\pi(x,\beta) = 1/\beta$; thus, maximizing $D_\pi(x,\beta)$ is equivalent to minimizing $U_\pi(x,\beta)$.

Let $R(x) = \min\{R, |C_0(x)|\}$ denote the maximum number of components that can be under repair when the system is in state x . A deterministic policy π is specified by sets of the form $\pi(x) = \{\pi_i(x) \in C_0(x),\ 1 \leq i \leq R(x)\}$, for all states x ; where $\pi(x)$ denotes the set of components under repair when π is used and the system is in state x . We assume that components in $\pi(x)$ are ordered in a consistent way for all x in S , but for notational simplicity we do not define this order explicitly. Thus, $\pi_i(x)$ denotes a specific failed component on which a repairman is assigned under π when the system is in state x . Under policies in Π , repairmen are never left idle when there are enough failed components thus, $\pi(x) = C_0(x)$ for all states x such that $R > |C_0(x)|$. However, for states x such that $R < |C_0(x)|$ there is a choice for $\pi(x)$, and the problem is to choose these sets so as to maximize $A(\pi)$ or $D_\pi(x,\beta)$.

Finally, we assume that the system under consideration is *coherent*, i.e., we place the following restrictions on G and B (or equivalently, on ϕ). i) If $x \in G$ and $y \geq x$ (i.e., $y_i \geq x_i$, $1 \leq i \leq N$) then $y \in G$ and if

$x \in B$ and $y \leq x$ then $y \in B$. ii) For any component i, $1 \leq i \leq N$, there exists a state $x \in G$ such that state $(0_i, x) \in B$; where we use the notation: $(\delta_k, x) = (x_1, \ldots, x_{k-1}, \delta, x_{k+1}, \ldots, x_N)$, $\delta = 0, 1$.

We shall use the following notation and terminology from coherent structure theory; see Barlow and Proschan (1974). For any state $x \in S$ we define: $C_0(x) = \{i \mid x_i = 0\}$, $C_1(x) = \{i \mid x_i = 1\}$. Given any finite set C, $|C|$ will denote the number of elements in it. A state $x \in B$ such that $y \in G$ for any $y \geq x$, $y \neq x$, is called a cut; it corresponds to minimal set of components which by failing cause the failure of the system. The *size* of a cut x is the number of components in $C_0(x)$.

3. Asymptotically Optimal Policies. We first consider the expected discounted system operation time criterion and then, using the results obtained, we treat the maximum availability problem.

By conditioning on the first transition out of state x we obtain that under a deterministic policy π the $U_\pi(x,\beta)$'s, $x \in S$, are the unique solution to the following system of linear equations

$$(5) \quad U(x) = \frac{1}{\lambda_\pi(x) + \rho\mu(x) + \beta} \left[\bar{\phi}(x) + \sum_{i=1}^{R(x)} \lambda_{\pi_i}(x) U(1_{\pi_i}(x), x) + \rho \sum_{j=1}^{N} \mu_j x_j U(0_j, x) \right]$$

$x \in S$, where we set $\lambda_\pi(x) = \sum_{i=1}^{R(x)} \lambda_{\pi_i}(x)$, $\mu(x) = \sum_{j=1}^{N} \mu_j x_j$ and $U(1_j, 1) = 0$. It is known, Derman (1970), that there exists a deterministic policy π^* such that

$$(6) \qquad U_{\pi^*}(x,\beta) \leq U_\pi(x,\beta) \quad \forall\, x \in S, \quad \forall\, \pi \neq \pi^*.$$

Furthermore, since the state space is finite, it follows (Derman (1970) that there exists a $\beta_0 > 0$ and a deterministic policy π° such that

$$(7) \qquad U_{\pi^\circ}(x,\beta) \leq U_\pi(x,\beta) \quad \forall\, x \in S, \quad \forall\, \pi \neq \pi^\circ \text{ and } \forall\, \beta \in (0,\beta_0),$$

and

$$(8) \qquad \lim_{\beta\to 0} \beta\, U_{\pi^o}(x,\beta) = \sum_{x\in B} e_{\pi^o}(x) = A(\pi^o) \;, \quad \forall\, x \in S \;.$$

Note that under π^o the availability of the system is equal to $1 - \bar{A}(\pi^o)$. Thus, it follows from (7) and (8) that if a policy minimizes $U_\pi(x,\beta)$ for small values of β, then it maximizes the availability of the system.

We first prove the following

<u>Lemma 1.</u> For any $x \in S$, $\beta \in (0,\infty)$ and $\pi \in \Pi$, there exist power series expansions of $U_\pi(x,\beta)$, of the form

$$(9) \qquad U_\pi(x,\beta) = \sum_0^\infty U_\pi^{(\nu)}(x,\beta)\rho^\nu$$

<u>Proof.</u> We can write the system of equations (5) in the following form:

$$(10) \quad \mathcal{U}(x) = \frac{1}{\lambda_\pi(x) + \beta} \left[\bar{\phi}(x) + \sum_{i=1}^{R(x)} \lambda_{\pi_i}(x)\,\mathcal{U}(1_{\pi_i}(x),x) + \rho \sum_{j=1}^{n} \mu_j x_j (\mathcal{U}(0_j,x) - \mathcal{U}(x))\right]$$

$x \in S$. In matrix form (10) can be written as: $\mathcal{U} = a + B\mathcal{U} + \rho C\mathcal{U}$ where subscripts and arguments have been supressed and a, B, C are appropriately defined. It is obvious that B is triangular so that $(I-B)^{-1}$ exists and may be computed in a recursive fashion. Thus, $\mathcal{U} = (I-B)^{-1}a + \rho(I-B)^{-1}C\mathcal{U}$ or in more compact form:

$$(11) \qquad \mathcal{U} = b(\pi,\beta) + \rho\, Q(\pi,\beta)\,\mathcal{U}$$

It follows from (11) that for any $k \geq 1$, we have:

$$(12) \qquad U_\pi(x,\beta) = b(\pi,\beta) + \sum_1^k \rho^\nu (Q(\pi,\beta))^\nu b(\pi,\beta) + (Q(\pi,\beta))^\nu (U_\pi(x,\beta))\rho^{k+1}$$

Now let $||Q(\pi,\beta)||$ denote a norm of the matrix $Q(\pi,\beta)$, then (12) implies that (9) holds for all $\rho \in (0, 1/||Q(\pi,\beta)||)$.

The next corollary provides a method for computing the coefficients $U_\pi^{(\nu)}(x,\beta)$ recursively for increasing $|C_0(X)|$. For notational simplicity we set $U_\pi^{(\nu)}(1_j,1) = 0$.

<u>Corollary 1.</u> For any $x \in S$, $\beta \in (0,\infty)$ and $\pi \in \Pi$, the $U_\pi^{(\nu)}(x,\beta)$'s can be computed recursively from equations (13), (14) below

$$(13a) \qquad U_\pi^{(0)}(1,\beta) = 0 \quad ,$$

$$(13b) \qquad U_\pi^{(0)}(x,\beta) = \frac{1}{\lambda_\pi(x) + \beta}\left[\bar{\phi}(x) + \sum_{i=1}^{R(x)} \lambda_{\pi_i}(x) U_\pi^{(0)}((1_{\pi_i(x)},x),\beta)\right] \quad ,$$

$$(14) \qquad U_\pi^{(\nu+1)}(x,\beta) = \frac{1}{\lambda_\pi(x) + \beta}\left[\sum_{i=1}^{R(x)} \lambda_{\pi_i}(x) U_\pi^{(\nu+1)}((1_{\pi_i(x)},x),\beta) + \sum_{j=1}^{N} \mu_j x_j (U_\pi^{(\nu)}((0_j,x),\beta) - U_\pi^{(\nu)}(x,\beta))\right],$$

$\nu \geq 0$, $x \in S$.

<u>Proof.</u> It follows from (11) that $U_\pi^{(0)} = (I - B)^{-1}a$ and since B is triangular, $U_\pi^{(0)}$ can be computed recursively by $U_\pi^{(0)} = a + BU_\pi^{(0)}$ which is (13). Similarly, $U_\pi^{(\nu+1)} = (I - B)^{-1}CU^{(\nu)}$ thus, $U_\pi^{(\nu+1)} = BU_\pi^{(\nu+1)} + CU^{(\nu)}$ which is (14).

<u>Remark 1.</u> Note that equations (13), (14) constitute a Gauss Seidel iteration method for solving the system of linear equations (5) for a specific choice of initial points. Thus the overall approach of determining policies that minimize the leading coefficients is essentially equivalent to employing a so called pre-Gauss Seidel iteration for the under consideration Markovian decision problem, see Thomas <u>et al</u>., (1984) and references given there.

We next aim to determine the leading coefficients of the power series (9).

We first need to define the following quantities. Let

$m(\phi) = \min\{|C_0(x)| \mid x \in B\}$, $B_{m(\phi)} = \{x \in B \mid |C_0(x)| = m(\phi)\}$ and $I(x) = \min\{|C_0(y)| \mid y \leq x , y \in B\} - |C_0(x)|$. In the terminology of coherent structure theory, $m(\phi)$ is the size of a cut state of minimal size, $B_{m(\phi)}$ is the set of all such states and $I(x)$ is the minimum number of components that must fail when the system is in state x , in order to cause a system failure. The next lemma summarizes properties of $I(x)$ that are easily verifiable from its definition.

<u>Lemma 2.</u> For any state x the following are true.

(i) $I(x) \leq m(\phi) = I(1)$.

(ii) $I(0_i,x) \geq I(x) - 1$, $\forall\, i \in C_1(x)$.

(iii) If $\phi(x) = 1$ then $I(x) \geq 1$.

(iv) If $y \in B_{m(\phi)}$, $k_i \in C_0(y)$ $i=1,2,\ldots\nu$ and $x = (1_{k_1},\ldots,1_{k_\nu},y)$ then $I(x) = \nu$.

We can now prove the following

<u>Lemma 3.</u> For any $x \in S$, any $\pi \in \Pi$ and for any $k = 0,1,\ldots m(\phi) - 1$ if $I(x) \geq k + 1$ then $U_\pi^{(k)}(x,\beta) = 0$.

<u>Proof.</u> By induction on k and subinduction on $|C_0(x)|$.

i) For $k = 0$, i.e. $I(x) \geq 1$ we have:

a) if $|C_0(x)| = 0$ then $x = 1$ and therefore $U_\pi^{(0)}(1,\beta) = 0$ by lemma 1 ,

b) assume that the lemma holds for all x such that $|C_0(x)| = \nu$. Then (13), the induction hypothesis, and the observation that: $\bar{\phi}(x) = 0$ when $I(x) \geq 1$ and $|C_0(1_{\pi_i}, x)| = \nu$ when $|C_0(x)| = \nu + 1$, imply that the lemma holds for all x such that $|C_0(x)| = \nu + 1$.

ii) Assume that the lemma is true for $k = k_0 < m(\phi) - 1$. We next show that it holds for $k = k_0 + 1$. Then:

a) If $|C_0(x)| = 0$, i.e., $x = 1$ then (14) becomes:

$$U_\pi^{(k_0+1)}(1,\beta) = \frac{1}{\beta}\left[\sum_{j=1}^{N}\mu_j(U_\pi^{(k_0)}((0_j,1),\beta) - U_\pi^{(k_0)}(1,\beta))\right]$$

Note now that $I(0_j,1) \geq I(1) - 1 \geq k_0$, by lemma 2 (ii) . Thus, the result follows since by the induction hypothesis (ii) we have that:
$U_\pi^{(k_0)}((0_j,1),\beta) = U_\pi^{(k_0)}(1,\beta)) = 0$.

b) Assume that the lemma holds, for any x such that $|C_0(x)| = \nu$. Consider a state x such that $|C_0(x)| = \nu + 1$. Then $|C_0(1_{\pi_i(x)},x)| = \nu$, and the induction hypothesis (b) implies that: $U_\pi^{(k_0+1)}((1_{\pi_i(x)},x),\beta) = 0$. Notice also that $I(0_j,x) \geq k_0$, $I(x) \geq k_0 + 1 \geq k_0$, thus induction hypothesis (ii) implies that $U_\pi^{(k_0)}((0_j,x),\beta) = U_\pi^{(k_0)}(x,\beta) = 0$. Now it is easy to complete the induction step using (14) .

A consequence of Lemmas 1 and 3 is the next

<u>Theorem 1.</u> For any $x \in S$ and any $\pi \in \Pi$, there exist constants $U_\pi^{(I(x))}(x,\beta)$ in $(0,\infty)$, such that:

(15) $U_\pi(x,\beta) = U_\pi^{(I(x))}(x,\beta)\rho^{I(x)} + 0(\rho^{I(x)})$

where the $U_\pi^{(I(x))}(x,\beta)$'s can be determined recursively as follows.

i) For all states x such that $I(x) = 0$,

$$U_\pi^{(0)}(x,\beta) = \frac{1}{\lambda_\pi(x) + \beta}\left[\bar{\phi}(x) + \sum_{i=1}^{R(x)}\lambda_{\pi_i(x)}U_\pi^{(0)}((1_{\pi_i(x)},x),\beta)\right] . \tag{16}$$

ii) For all x such that $I(x) \geq 1$,

$$U_\pi^{(I(x))}(x,\beta) = \frac{1}{\lambda_\pi(x) + \beta}\left[\sum_{i=1}^{R(x)}\lambda_{\pi_i(x)}U_\pi^{(I(x))}((1_{\pi_i(x)},x),\beta) + \right.$$
$$\left. + \sum_{j=1}^{N}\mu_j x_j(U_\pi^{(I(x)-1)}((0_j,x),\beta)\right] \tag{17}$$

Proof. Equations (16) and (17) follow from corollary 1 and lemma 3. To show that $U_{\pi}^{(I(x))}(x,\beta) \geq 0$ for all x, notice that this is true for all x such that $I(x) = 0$. The proof can be completed by induction on $I(x)$ using (17).

Theorem 1 shows that the order of the leading terms in the asymptotic power series expansion of $U_{\pi}(x,\beta)$ is the same for all deterministic policies. Thus, we can formally state the following.

Proposition 1. A policy $\pi^* \in \Pi$ maximizes the expected total discounted system operation time for small values of ρ if and only if:

$$(18) \quad U_{\pi^*}(x,\beta) = \min\{ U_{\pi}^{(I(x))}(x,\beta) \; , \; \pi \in \Pi \} \; , \quad \forall\, x \in S \; .$$

In the absence of ties (18) determine unique asymptotically optimal actions for all states. Ties can be resolved by computing and minimizing higher order coefficients subject to minimization of all lower order coefficients.

Remark 2. Since $U_{\pi}^{(k)}(1,\beta) = 0$ for $k = 0, 1, \ldots, I(1) - 1$, the coefficients $U_{\pi}^{(k)}(x,\beta)$, for all $x \neq 1$, $k \leq I(x) - 1$, are the same with those in the asymptotic power series expansion of the expected discounted time that the system spends in failed states during the first passage time from state x to state 1 under policy π. Thus, since $I(x) \leq I(1) - 1$ for all $x \neq 1$, we have the following partial characterization of asymptotically optimal policies. If a policy is asymptotically optimal with respect to the expected discounted system operation time criterion then, it must assign repairmen to failed components in such a way that the expected discounted time that the system spends in failed states during the first passage time from state x to state 1 is minimized.

We now turn to the problem of determining $\pi \in \Pi$ to maximize the availability of the system. We have seen that for this problem it suffices to determine policies that minimize the leading coefficients $U_\pi^{(\nu)}(x,\beta)$ of the power series (9) for small values of the discount rate β. Thus, let $V_\pi^{(\nu)}(x) = \lim_{\beta \to 0} U_\pi^{(\nu)}(x,\beta)$, for $x \neq 1$ and $\nu \leq I(1) - 1$. Then, a policy π minimizes $U_\pi^{(\nu)}(x,\beta)$ for small for small values of β if and only if it minimizes $V_\pi^{(\nu)}(x)$. Now, using lemma 3 and theorem 1 we obtain the following procedure for computing and minimizing the $V_\pi^{(\nu)}(x)$'s directly.

<u>Theorem 2.</u> For any $x \in S$, $x \neq 1$ and any $\pi \in \Pi$, we have

(19) $V_\pi^{(\nu)}(x) = 0$ for $\nu \leq I(x) -1$,

and, the $V_\pi^{(I(x))}(x)$'s can be determined recursively as follows.

i) For all states x such that $I(x) = 0$,

$$(20) \qquad V_\pi^{(0)}(x) = \frac{1}{\lambda_\pi(x)} \left[\bar{\phi}(x) + \sum_{i=1}^{R(x)} \lambda \pi_i(x) V_\pi^{(0)}((1\pi_i(x),x)) \right].$$

ii) For all x such that $I(x) \geq 1$,

$$(21) \qquad V_\pi^{(I(x))}(x) = \frac{1}{\lambda_\pi(x)} \left[\sum_{i=1}^{R(x)} \lambda \pi_i(x) V_\pi^{(I(x))}((1\pi_i(x),x)) + \sum_{j=1}^{N} \mu_j x_j (V_\pi^{(I(x)-1)}((0_j,x)) \right].$$

iii) For $I(x) \leq \nu < I(1) - 1$,

$$(22) \qquad V_\pi^{(\nu+1)}(x) = \frac{1}{\lambda_\pi(x)} \left[\sum_{i=1}^{R(x)} \lambda \pi_i(x) V_\pi^{(\nu+1)}((1\pi_i(x),x)) + \sum_{j=1}^{N} \mu_j x_j (V_\pi^{(\nu)}((0_j,x)) - V_\pi^{(\nu)}(x)) \right].$$

Thus we can now formaly state the following

<u>Proposition 2.</u> A policy $\pi \in \Pi$ maximizes the availability of the system, for small values of ρ if and only if :

$$(23) \quad V_{\pi^*}^{(I(x))}(x) = \min\{ V_{\pi}^{(I(x))}(x) \ , \ \pi \in \Pi \} \ , \quad \forall \ x \in S \ , \ x \neq 1 \ .$$

In the absence of ties, (23) determines a unique asymptotically optimal policy. Ties can be resolved by considering higher order coefficients as computed by (22).

<u>Remark 3.</u> Notice that when the system is in a failed state x, $I(x) = 0$ and the $V_{\pi}^{(0)}(x)$ as determined from equations (20) is the expected time until the system is back in operation, in the absence of failures. Thus, we obtain the following, intuitively expected, partial characterization of asymptotically optimal policies. If a policy π^o maximizes the availability of the system, for small values of ρ, then when the system is failed, π^o must assign repairmen to failed components in such a way that the expected time until the system is back in operation, in the absence of failures, is minimized.

In the next theorem we show that asymptotically optimal policies are strictly optimal when all failure rates are sufficiently small.

<u>Theorem 3.</u> Let π^* be an asymptotically optimal policy, with respect to one of the criteria that have been considered. Then, there exists a $\rho_0 > 0$ such that π^* is optimal $\forall \ \rho \in (0,\rho_0)$.

<u>Proof.</u> We prove the theorem for the expected discounted operation time criterion only. The proof for the maximum availability criterion is similar and is omitted.

Recall that for any policy $\pi \in \Pi$ and for $\rho \in (0, 1/||Q(\pi)||)$ the $U_\pi(x,\beta)$'s possess convergent power series expansions. Since, there are finite many policies in Π, it follows that the above power series represent-

ations of all $U_\pi(x,\beta)$'s are convergent for all $\pi \in \Pi$ in the interval $(0,\rho_1)$, where $\rho_1 = \min_{\pi\in\Pi} \{ 1/||Q(\pi,\beta)|| \}$.

Now for any $x \in S$ and π_1 , $\pi_2 \in \Pi$, it follows (see Rudin (1976, pp. 177)) that the difference: $U_{\pi_1}(x,\beta) - U_{\pi_2}(x,\beta)$ may change sign a finite number of times. Thus the theorem follows from Proposition 2 and the fact that there are finite many policies in Π and states in S .

4. Applications. In the following examples we restrict our attention to determining policies which are asymptotically optimal with respect to the availability criterion.

4.1 Series and Parallel Systems. Consider first the N component series system maintained by R repairmen. The only functioning state is state $1 = (1,\ldots,1)$. From Proposition 2, remark 3 we know that an asymptotically optimal policy π^* minimizes the expected time to state 1 from any initial state x in the absence of failures. Thus for any initial state x , in the terminology of stochastic scheduling, an asymptotically optimal policy minimizes the expected makespan for allocating $|C_0(x)|$ tasks (repairs) on R identical processors (repairmen). For $R = 1$ all policies in Π have the same makespan: $V_\pi^{(0)}(x) = \sum_{i\in C_0(x)} 1/\lambda_i$. For $R \geq 2$ it has been shown by Bruno et al. (1981) that an optimal policy assigns repairmen to failed components in $C_0(x)$ according to the LEPT (Longest Expected Processing Time First) rule. In the context of the series system an asymptotically optimal policy assigns repairmen to the failed components with the longest expected repair times. Notice now that this LEPT policy is optimal for sufficiently small failure rates (Theorem 3). It follows from this example that in the general case the optimal policy does depend on the repair rates and therefore the result established in Katehakis and Derman (1984) does not hold for $R \geq 2$.

For the parallel system the only failed state is state $\mathbf{0} = (0,\ldots,0)$. Furthermore, $I(1_j,x) = I(x) + 1$ for all $x \neq 1$, thus it is easy to show, using Theorem 2, that the policy which always assigns repairmen to the failed components with the smallest repair rates is asymptotically optimal.

<u>4.2 Parallel Subsystems Connected in Series.</u> Consider a system that is composed of K subsystems and it is maintained by a single repairman. The i^{th} subsystem is composed of N_i components with the same failure rates μ_i . Furthermore, we assume that all components have identical repair rates λ .

Since the subsystems have identical components, it is easy to see that the state of the system at any time can be adequately specified by a vector $z = (z_1,\ldots,z_K)$, where z_i denotes the number of functioning components in subsystem i , $z_i = 0,1,\ldots N_i$. The structure of the system is specified by the sets $G = \{z \mid z_i \geq 1 \ \forall\ i\}$, $B = \{z \mid z_i = 0 \text{ for some } i\}$. Let $C_0(z) = \{i \mid z_i = 0\}$, $(n_j,z) = (z_1,\ldots,z_{j-1},n,z_{j+1},\ldots,z_K)$, $n = 0,1,\ldots,N_j$, and define $(n_{j_1},\ldots,n_{j_k},z)$ recursively by $(n_{j_1},n_{j_2},z) = (n_{j_1},(n_{j_2},z))$. Note that $I(z) = \min\{z_i , i = 1,\ldots,K\}$. Finally let $\mathcal{I}(z) = \{i \mid z_i = I(z)\}$ and $J((n_{j_1},\ldots, n_{j_k},z)) = \min\{z_i , i \neq j_1,\ldots,j_k\}$. Since subsystems have identical components a policy is specified up to the subsystem on which the repairman is assigned only. With this generalization of notation, using Theorem 2, we obtain that the asymptotically optimal policy is given by the following simple rule.

i) If $z \in G$ assign the repairman to system j if and only if either $\mathcal{I}(z) = \{j\}$ or $\mu_j = \max\{\mu_i , i \in \mathcal{I}(z)\}$.

ii) If $z \in B$ assign the repairman to subsystem j if and only if $\mu_j = \min\{\mu_i , i \in C_0(z)\}$.

The proof of i) essentially involves establishing by induction on n that

if $J((n_{i_1},\ldots,n_{i_k},z)) \geq k+1$ then,

$$(24) \qquad V_{\pi 0}^{(n)}(n_{i_1},\ldots,n_{i_k},z) = n! \sum_{j=1}^{n} j\mu_{i_j}^{n} \lambda^{-(n+1)}, \; n = 1,\ldots,I((N_1,\ldots,N_K)) - 1.$$

The proof of ii) is easy to complete by induction on decreasing $|J(z)|$.

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